Math 1010C Term 1 2015
Supplementary exercises 7

1. (Putnam 2002) Let k be a fixed positive integer. The n-th derivative of $\dfrac{1}{x^k-1}$ has the form $\dfrac{P_n(x)}{(x^k-1)^{n+1}}$ where $P_n(x)$ is a polynomial. Find $P_n(1)$ for all $n \geq 0$.

   (Hint: Use induction on n. More precisely, by differentiating $\dfrac{P_n(x)}{(x^k-1)^{n+1}}$, find a recurrence relation between $P_{n+1}(1)$ and $P_n(1)$. Answer $= (-k)^n n!$.)

2. (Putnam 1998) Find the minimum value of

$$\frac{\left(x+\frac{1}{x}\right)^6-\left(x^6+\frac{1}{x^6}\right)-2}{\left(x+\frac{1}{x}\right)^3+\left(x^3+\frac{1}{x^3}\right)}$$

   for $x > 0$. (Hint: The substitution $t = x + \dfrac{1}{x}$ would work, but it is faster to observe that in the numerator, we have

$$x^6+\frac{1}{x^6}+2=\left(x^3+\frac{1}{x^3}\right)^2,$$

   so that the numerator is just A^2-B^2 if we write $A = \left(x+\frac{1}{x}\right)^3$ and $B = x^3+\frac{1}{x^3}$.)

3. (Putnam 1998) Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that

$$f(a)\cdot f'(a)\cdot f''(a)\cdot f'''(a) \geq 0.$$

   (Hint: We would be done if any of the functions f, f', f'', f''' has a zero. So by continuity, we may assume that each of these four functions is either strictly positive, or strictly negative. Without loss of generality, we may assume $f'' > 0$ and $f''' > 0$ (why?). In this case, f' is strictly increasing and strictly convex, so $f'(x)$ must be positive for large enough x. As a result, f is strictly increasing and strictly convex on the half line $[b, \infty)$ when b is sufficiently large. Hence $f(x)$ must also be positive for large enough x.)

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Supplementary exercises 9

1. Compute
$$\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{\arctan(x^2)}{1+x^2} dx.$$
Hint: Substitute $u = \frac{1}{x}$, and show that if I is the integral above, then
$$I = \frac{\pi}{2} \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{1}{1+x^2} dx - I.$$

2. Compute
$$\int \frac{1}{1+\sin x} dx.$$
Hint: Multiply by $\dfrac{1-\sin x}{1-\sin x}$ (or use t-substitution, i.e. substitute $t = \tan \dfrac{x}{2}$)

3. Compute
$$\int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan x)^{\sqrt{5}}} dx.$$
Hint: Substitute $u = \frac{\pi}{2} - x$, and use symmetry.

4. Compute
$$\int \cos(\ln x) dx.$$
Hint: Integrate by parts twice, or use the substitution $u = \ln x$ to reduce to the more familiar integral $\int e^x \cos x dx$.

5. Compute
$$\int \frac{\sin x}{\sin x + \cos x} dx$$
Hint: Either let
$$I = \int \frac{\sin x}{\sin x + \cos x} dx, \quad J = \int \frac{\cos x}{\sin x + \cos x} dx$$
and compute both $I + J$ and $I - J$ (the latter can be computed using the substitution $u = \sin x + \cos x$), or write
$$\frac{\sin x}{\sin x + \cos x} = \frac{\tan x}{1 + \tan x}$$
and substitute $u = \tan x$. Yet another way is t-substitution: substitute $t = \tan \frac{x}{2}$.

6. Compute

$$\int \frac{1}{\sin(x-a)\sin(x-b)}dx$$

if $a-b$ is not a multiple of π.
Hint:

$$\int \frac{1}{\sin(x-a)\sin(x-b)}dx = \frac{1}{\sin(a-b)}\int \frac{\sin[(x-b)-(x-a)]}{\sin(x-a)\sin(x-b)}dx.$$

7. Compute

$$\int \sqrt{1+e^{2x}}dx.$$

Hint: Substitute $u = \sqrt{1+e^{2x}}$. Or else, substitute $v = e^x$. Then one arrives at the integral

$$\int \frac{\sec^3 x}{\tan x}dx,$$

which one could integrate by using

$$\int \frac{\sec^3 x}{\tan x}dx = \int \frac{\sec^3 x \tan x}{\tan^2 x}dx = \int \frac{\sec^2 x}{\sec^2 x - 1}d(\sec x).$$

The integrand in the last integral is a rational function in $\sec x$, which can be integrated using partial fractions.

8. Compute

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}dx$$

Hint: Let I be the above integral. Substitute $x = \tan\theta$ to see that

$$I = \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)d\theta.$$

Substitute $t = \frac{\pi}{4} - \theta$ to see that

$$I = \ln 2 \int_0^{\frac{\pi}{4}} dt - I$$

so $I = \frac{\pi}{8}\ln 2$.

9. Compute

$$\int_0^1 (1-x^7)^{1/5} - (1-x^5)^{1/7}dx.$$

Hint: The answer is 0; one just needs to show

$$\int_0^1 (1-x^7)^{1/5}dx = \int_0^1 (1-y^5)^{1/7}dy.$$

But both are the area bounded by the curve $x^7 + y^5 = 1$ with the x and y-axes. So the two integrals are equal.